Accelerating Neural Network Training using Arbitrary Precision Approximating Matrix Multiplication Algorithms

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Summary of Contributions

- Curate practical, approximate, fast matrix multiplication algorithms
- Generate efficient multithreaded code for all of them, with a 25% improvement over best classical implementation (BLAS GEMM)
- Demonstrate robustness of Neural Network training with approximate matrix multiplication
- Accelerate multithreaded training performance of Multi-Layer Perceptron and VGG-19 networks up to 15%

Neural Network

Application:

Face Recognition

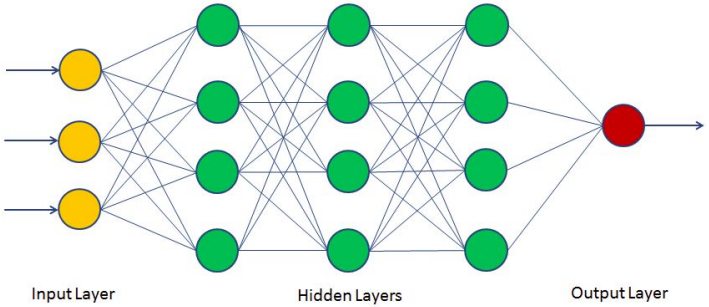
Smart Assistant

Related Company: Amazon, Google, etc

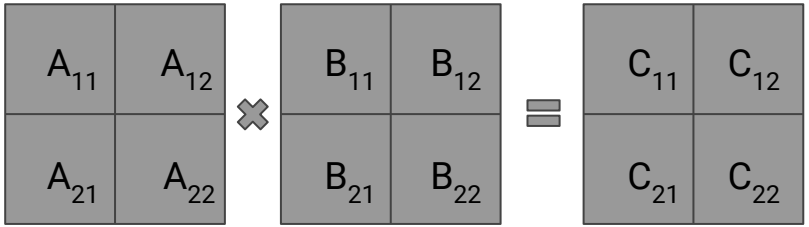
Theoretical Basis:

Weight Optimization

Matrix Operation: Matrix Multiplication



Matrix Multiplication



A B C

$C_{11} = A_{11}*B_{11} + A_{12}*B_{21}$

$C_{21} = A_{21}*B_{11} + A_{22}*B_{21}$

$C_{12} = A_{11}*B_{12} + A_{12}*B_{22}$

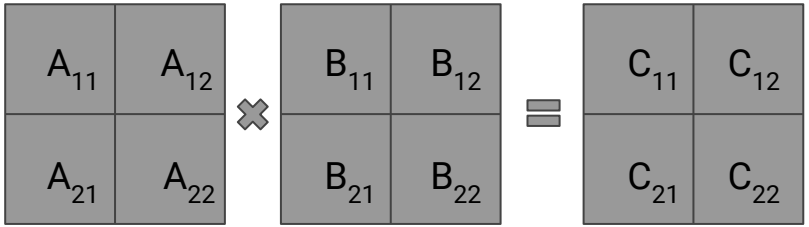
$C_{22} = A_{21}*B_{12} + A_{22}*B_{22}$

Classic Algorithm:

8 multiplications

Complexity: $O(n^3)$

Strassen's Algorithm



A B C

7 multiplications
less subproblems per iteration

$\log_2 7 \approx 2.81$

Complexity: $O(n^{2.81})$

$C_{11} = (A_{11}+A_{22})(B_{11}+B_{22})+A_{22}(B_{21}-B_{11})-(A_{11}+A_{12})B_{22}+(A_{12}-A_{22})(B_{21}+B_{22})$

$C_{12} = A_{11}(B_{12}-B_{22})+(A_{11}+A_{12})B_{22}$

$C_{21} = (A_{21}+A_{22})B_{11}+A_{22}(B_{21}-B_{11})$

$C_{22} = (A_{11}+A_{22})(B_{11}+B_{22})-(A_{21}+A_{22})B_{11}+A_{11}(B_{12}-B_{22})+(A_{21}-A_{11})(B_{11}+B_{12})$

Matrix Dimension

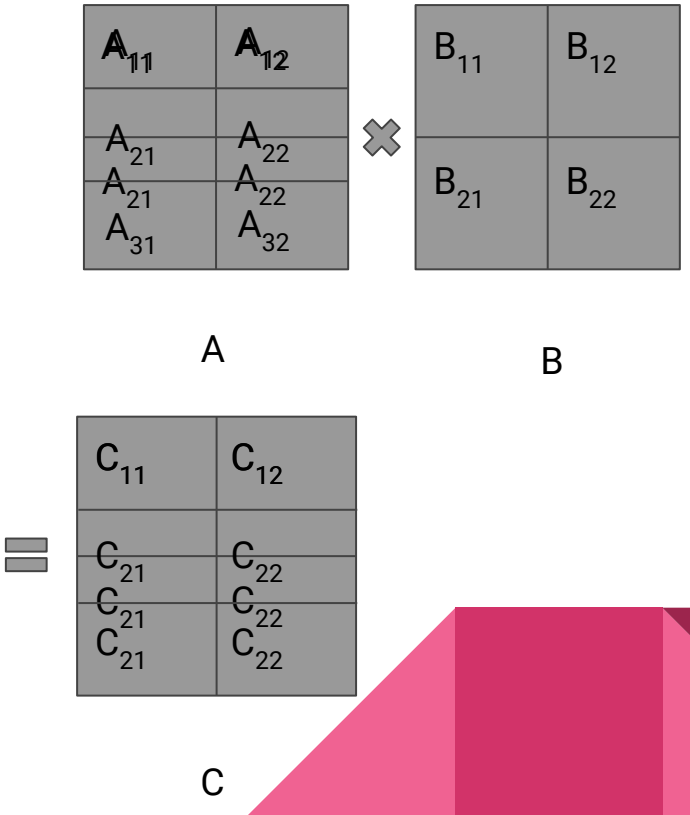
Strassen: $\log_{2x2x2} 7^3 \approx 2.81$

Bini: $\log_{3x2x2} 10^3 = \log_{12} 1000 \approx 2.78$

Algorithm shape/dimension:

Strassen <2,2,2;7>

Bini <3,2,2;10>



APA -- Arbitrary Precision Approximation

Bini's 322 Algorithm <3,2,2>:

$$C_{11} = ((A_{11}+A_{22})\cdot(\varepsilon B_{11}+B_{22})+A_{22}\cdot(-B_{21}-B_{22})-A_{11}\cdot B_{22}+(\varepsilon A_{12}+A_{22})\cdot(-\varepsilon B_{11}+B_{21}))/\varepsilon$$

$$C_{12} = (-A_{11}\cdot B_{22}+(A_{11}+\varepsilon A_{12})\cdot(\varepsilon B_{12}+B_{22}))/\varepsilon$$

$$C_{21} = (\varepsilon A_{12}+A_{22})\cdot(-\varepsilon B_{11}+B_{21})+(A_{21}+A_{32})\cdot(B_{11}+\varepsilon B_{22})-(\varepsilon A_{31}+A_{32})\cdot(B_{12}-\varepsilon B_{22})$$

$$C_{22} = (A_{11}+A_{22})\cdot(\varepsilon B_{11}+B_{22})-(A_{11}+\varepsilon A_{12})\cdot(\varepsilon B_{12}+B_{22})+(A_{21}+\varepsilon A_{31})\cdot(B_{12}-\varepsilon B_{22})$$

$$C_{31} = (-A_{32}\cdot B_{11}+(\varepsilon A_{31}+A_{32})\cdot(B_{12}-\varepsilon B_{22}))/\varepsilon$$

$$C_{32} = ((A_{21}+A_{32})\cdot(B_{11}+\varepsilon B_{22})+A_{21}\cdot(-B_{11}-B_{12})-A_{32}\cdot B_{11}+(A_{21}+\varepsilon A_{31})\cdot(B_{12}-\varepsilon B_{22}))/\varepsilon$$

10 multiplication on (3, 2) x (2, 2) matrices $\longrightarrow O(n^{2.78})$

ε: approximation parameter

Roundoff Error
Approximation Error

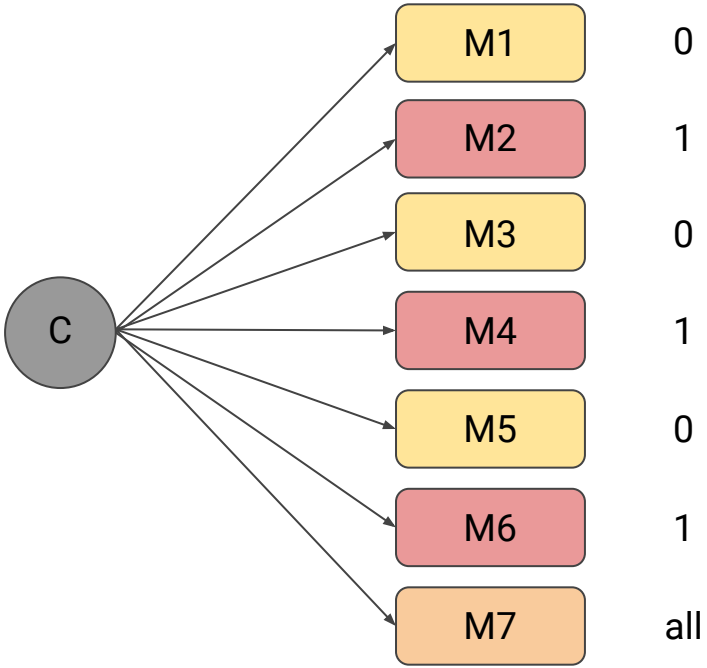
12 APA algorithms

Ref	Dimension	Relative Error	Complexity	Speedup
[1]	<3,2,2>	1e-3	$n^{2.78}$	20%
[2]	<3,3,3;21>	1e-2	$n^{2.771}$	28.6%
[3]	<4,2,2>	1e-2	$n^{2.775}$	23.1%
[3]	<5,2,2>	1e-2	$n^{2.777}$	25%
[3]	<3,3,2>	1e-2	$n^{2.739}$	28.6%
[4]	<4,3,3>	1e-2	$n^{2.759}$	33.3%
[5]	<4,4,2>	1e-2	$n^{2.751}$	33.3%
[6]	<4,4,4>	1e-2	$n^{2.762}$	39.1%
[7]	<5,5,2>	1e-2	$n^{2.769}$	35.1%
[6]	<5,5,5>	1e-2	$n^{2.796}$	38.9%
[8]	<7,2,2>	1e-1	$n^{2.783}$	27.3%
[9]	<3,3,3;20>	1e-1	$n^{2.727}$	35%

Single precision

Neural Networks' Robustness

Parallel Implementation

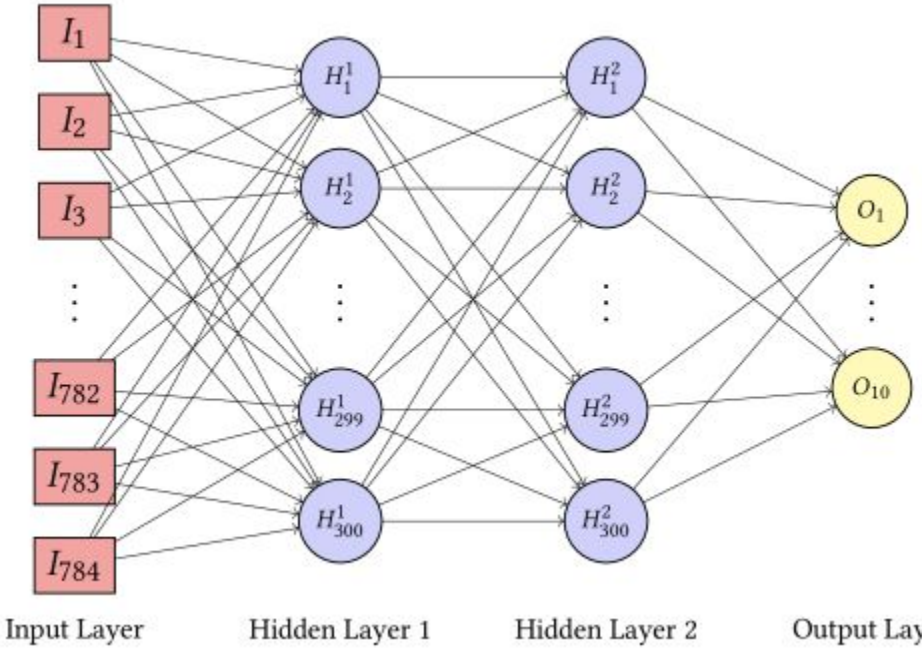


Hybrid approach: BFS + DFS

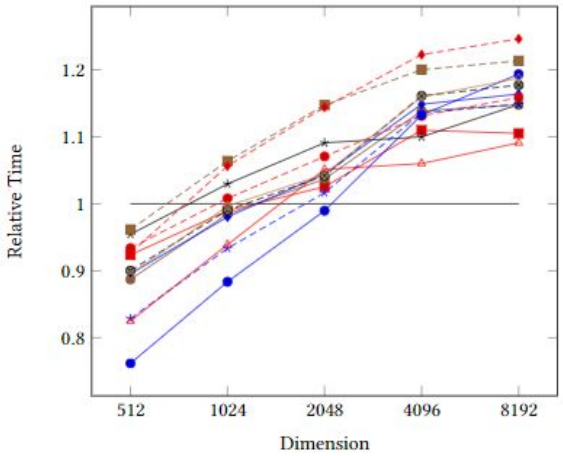
Strassen’s algorithm implementation on 2 threads

Code generation for each algorithm

Multi-Layer Perceptron

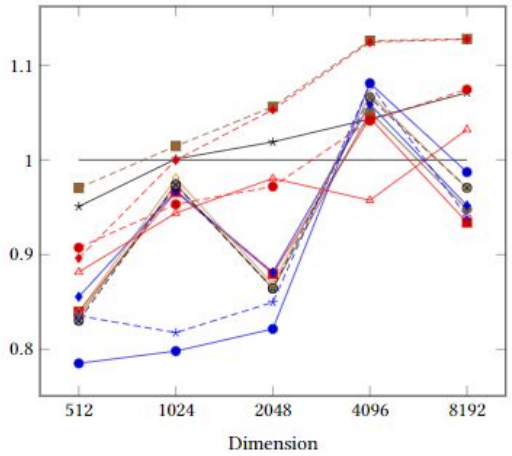


Parallel Performance on Neural Network Training



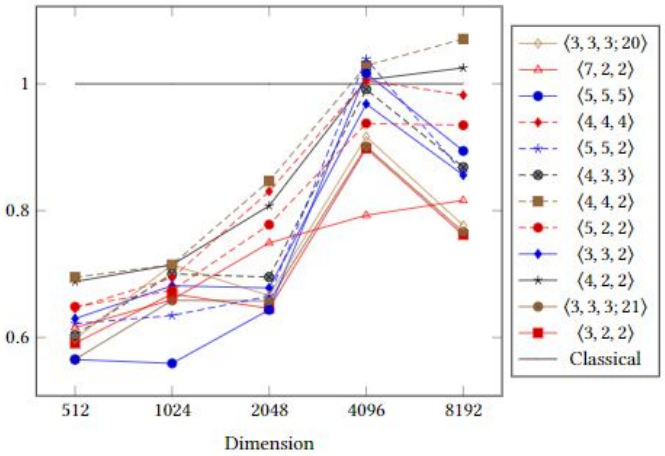
(a) One thread

8%~25%



(b) Six threads

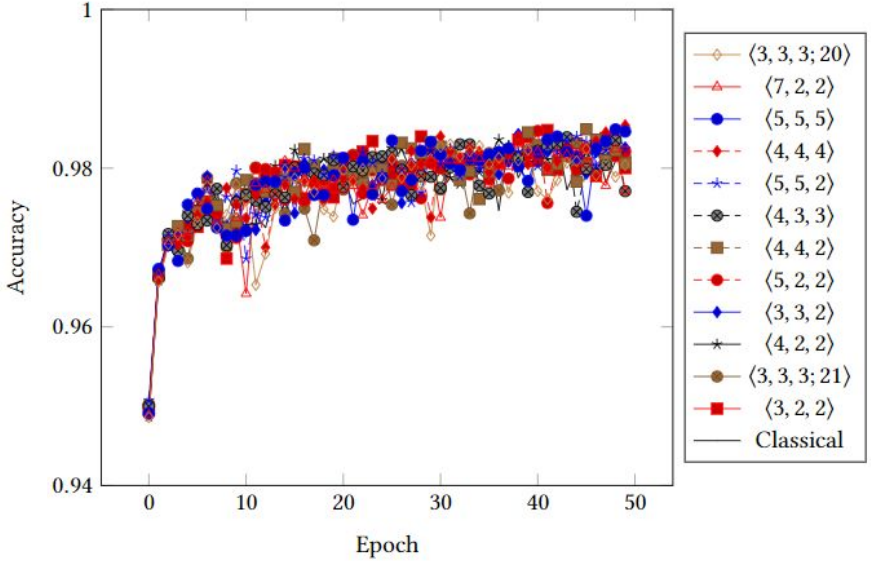
13%: <4,4,2>&<4,4,4>



(c) Twelve threads

7%: <4,4,2>

APA Algorithms Preserve Accuracy



97% ~99% Test accuracy

VGG-19 Model

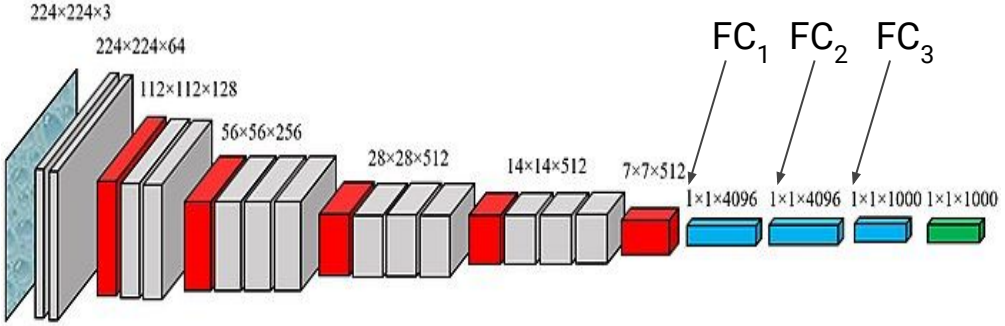
3 Fully-connected Layers:

FC_1 *25088* x *Batch Size* x *4096*
Batch Size x *25088* x *4096*
Batch Size x *4096* x *25088*

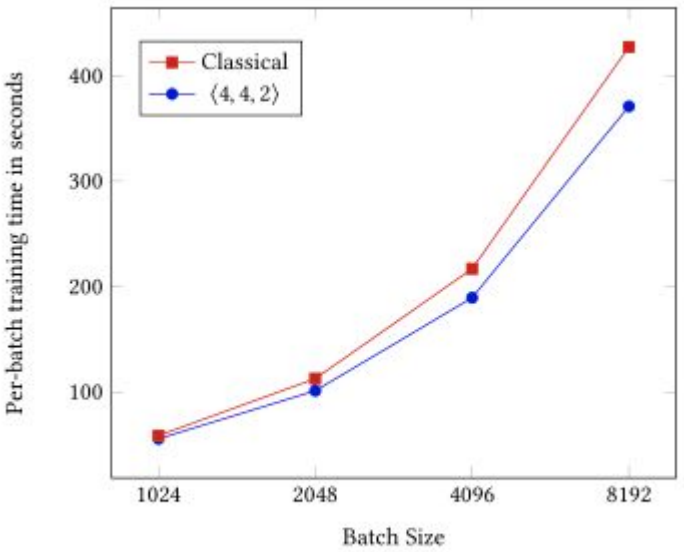
FC_2 *4096* x *Batch Size* x *4096*
Batch Size x *4096* x *4096*
Batch Size x *4096* x *4096*

FC_3 *4096* x *Batch Size* x *1000*
Batch Size x *4096* x *1000*
Batch Size x *1000* x *4096*

Batch Size ={1024, 2048, 3072, 4096}

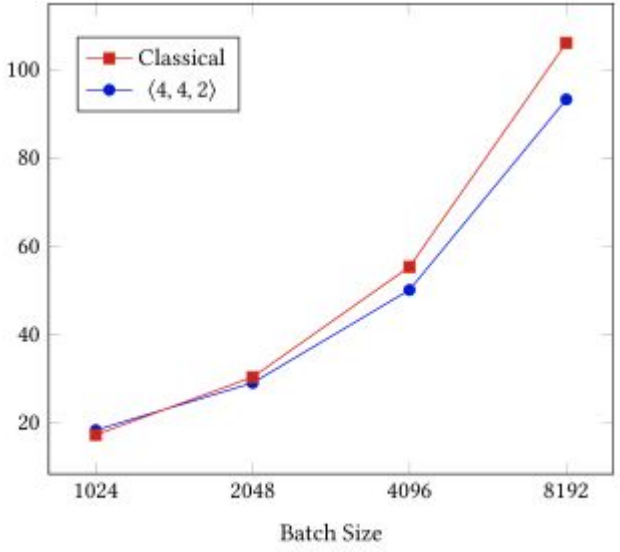


VGG-19 Model



(a) One thread

15% Speedup



(b) Six threads

10% Speedup

Summary of Contributions

- Curate practical, approximate, fast matrix multiplication algorithms
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- Demonstrate robustness of Neural Network training with approximate matrix multiplication
- Accelerate multithreaded training performance of Multi-Layer Perceptron and VGG-19 networks up to 15%
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Reference

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5. A.V. Smirnov. 2015. A bilinear algorithm of length 22 for approximate multiplication of 2 x 7 and 7 x 2 matrices. Computational Mathematics and Mathematical Physics
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Algorithm Numerical Accuracy

Algorithm	σ	φ	Relative error
Classic	/	/	1e-7
<3,2,2>	1	1	1e-3
<3,3,3;21>	1	2	1e-2
<4,2,2>	1	2	1e-2
<5,2,2>	1	3	1e-2
<3,3,2>	1	3	1e-2
<4,3,3>	1	3	1e-2
<4,4,2>	1	3	1e-2
<4,4,4>	1	3	1e-2
<5,5,2>	1	3	1e-2
<5,5,5>	1	3	1e-2
<7,2,2>	1	5	1e-1
<3,3,3;20>	1	6	1e-1

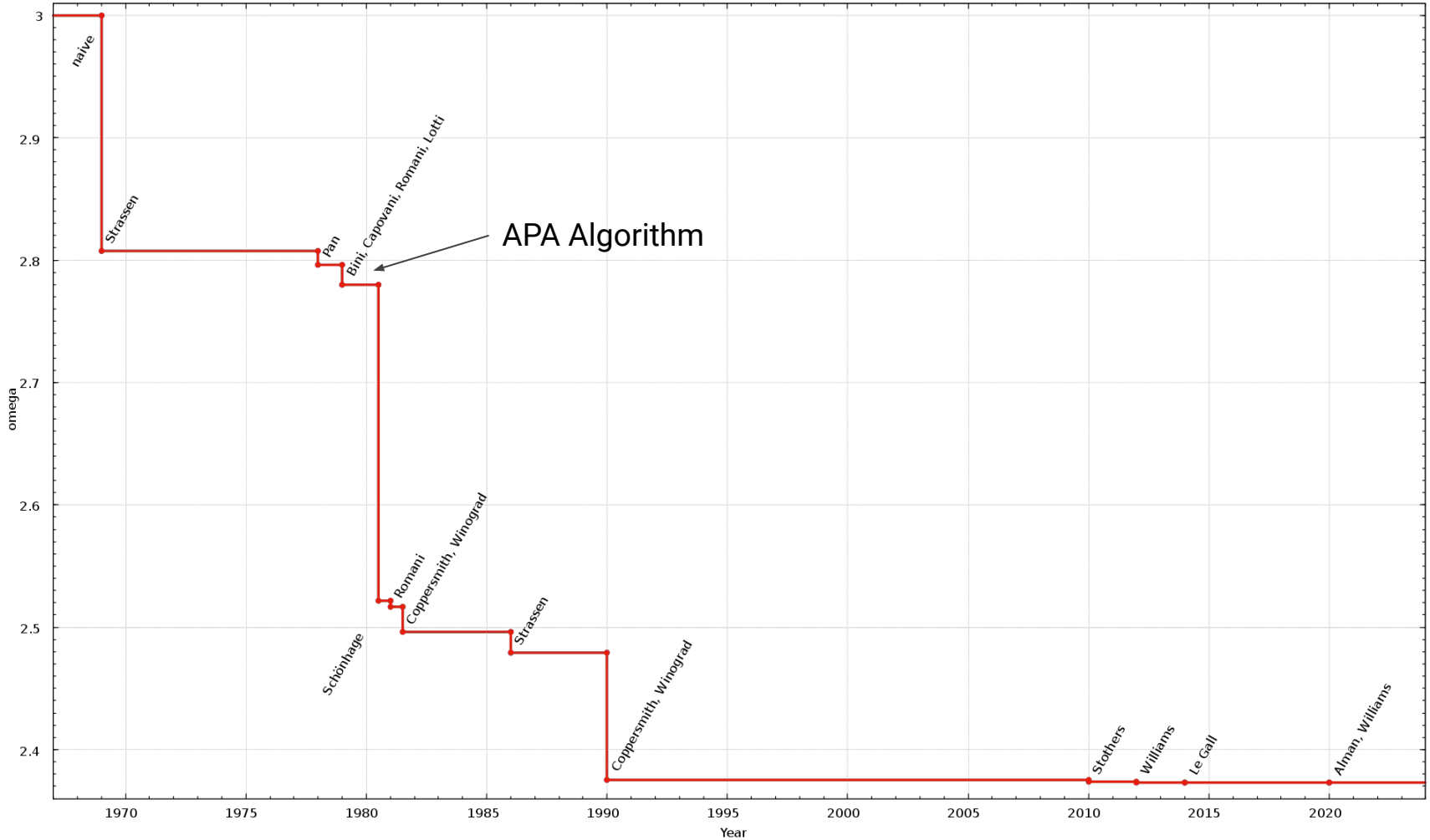
σ: Approximation error

φ: Roundoff error

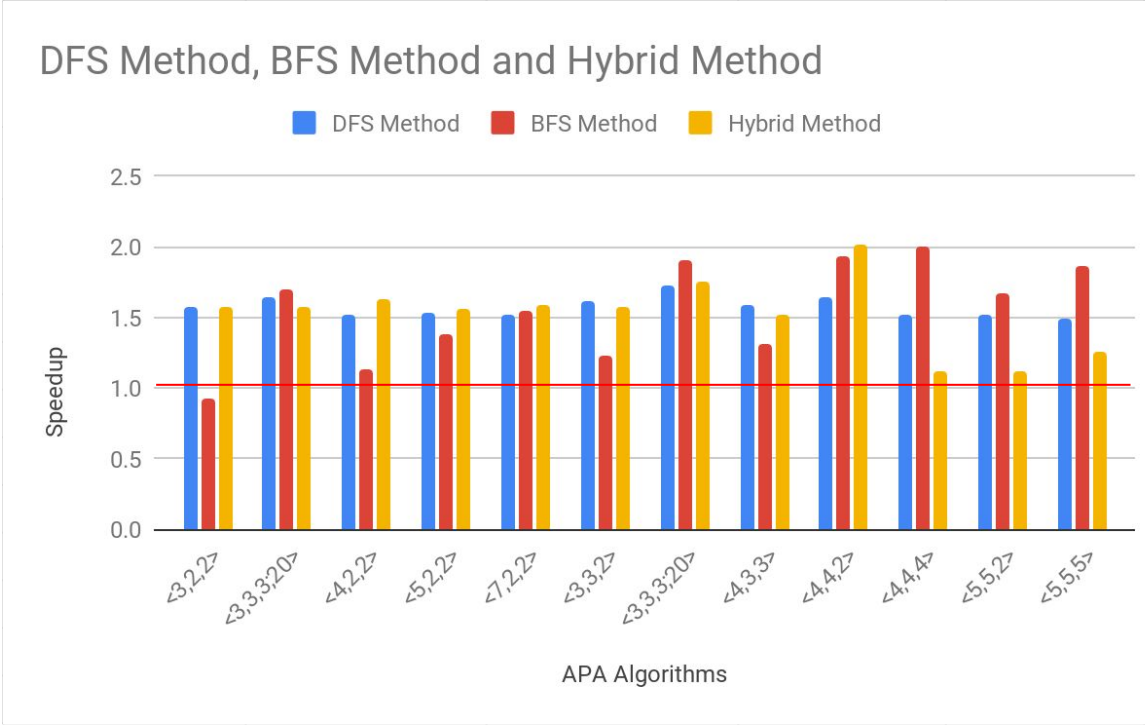
ε: Approximation Parameter

Bini's Theorem: The error produced by an APA-algorithm is $e = O(2^{-d\sigma/(\sigma+\varphi)})$, which is minimized when $\varepsilon = O(2^{-d/(\sigma+\varphi)})$.

2^{-d}: floating point precision


naïve
Strassen
Pan
Bini, Capovani, Romani, Lotti
APA Algorithm
Schönhage
Romani
Coppersmith, Winograd
Strassen
Coppersmith, Winograd
Stothers
Williams
Le Gall
Alman, Williams
omega
Year
3
2.9
2.8
2.7
2.6
2.5
2.4
1970
1975
1980
1985
1990
1995
2000
2005
2010
2015
2020

CPU Parallel Implementation



12-core CPU

Matrix Multiplication of size 6000x6000

12 APA Algorithms